NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1298

ON THE PROBLEM OF GAS FLOW OVER AN INFINITE CASCADE USING CHAPLYGIN'S APPROXIMATION

By G. A. Bugaenko

Translation

"K Voprosu o Struinom Obtekanii Beskonechnoi Reshetki Gazom v Priblizhennoi Postanovke S. A. Chaplygina." Prikladnaya Matematika i Mekhanika, T. XIII, No. 4, 1949.

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1. Some well-known results of Chaplygin's method (reference 1) are first presented. For the adiabatic law of the state of the gas when $p = k\rho^{\gamma}$, the following relations hold:

$$\left.\begin{aligned} p &= p_0\left(1 - \frac{V^2}{2\alpha}\right)^{\beta+1} \\ \rho &= \rho_0\left(1 - \frac{V^2}{2\alpha}\right)^{\beta} \end{aligned}\right\} \qquad (1.1)$$

where p is the gas pressure, ρ is the gas density, V is the modulus of velocity, p_0 and ρ_0 are the values of p and ρ at the critical point of the flow at which the velocity becomes zero, k is the coefficient of proportionality, γ is the ratio of specific heats, and α and β are constants.

If the angle θ between the velocity and the x-axis and the magnitude τ equal to $V^2/2\alpha$ are considered, then, as was shown by Chaplygin, the equations of gas motion assume the form

$$\left.\begin{aligned} \frac{\partial\tau}{\partial\psi} &= \frac{2\tau}{(1-\tau)^{\beta}}\frac{\partial\theta}{\partial\varphi} \\ \frac{\partial\tau}{\partial\varphi} &= -\frac{2\tau(1-\tau)^{\beta+1}}{1-(2\beta+1)\tau}\frac{\partial\theta}{\partial\psi} \end{aligned}\right\} \qquad (1.2)$$

where φ is the velocity potential and ψ is the stream function.

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*"K Voprosu o Struinom Obtekanii Beskonechnoi Reshetki Gazom v Priblizhennoi Postanovke S. A. Chaplygina." Prikladnaya Matematika i Mekhanika, T. XIII, No. 4, 1949, pp. 449 - 456.

Chaplygin reduced these equations to the very simple form

$$\left.\begin{aligned} \frac{\partial\sigma}{\partial\psi} &= \frac{\partial\theta}{\partial\varphi} \\ \frac{\partial\sigma}{\partial\varphi} &= -\frac{1}{K}\frac{\partial\theta}{\partial\psi} \end{aligned}\right\} \tag{1.3}$$

by introducing the new variable σ and the constant K defined by the formulas

$$\left.\begin{aligned} \sigma &= \int_{\tau_{\infty 2}}^{\tau} \frac{(1-\tau)^{\beta}}{2\tau}\, d\tau \\ K &= \frac{1-(2\beta+1)\tau}{(1-\tau)^{2\beta+1}} \end{aligned}\right\} \tag{1.4}$$

Chaplygin showed that for velocities far removed from the velocity of sound, the magnitude K is approximately unity and equations (1.3) can be integrated by assuming K equal to 1. For $K = 1$, these equations go over into the conditions of Cauchy-Riemann; hence, $\omega = \theta + i\sigma$ will be an analytical function of the complex variable $f = \varphi + i\psi$.

The equation for the elementary vector along a streamline in the approximate treatment has, as is known (reference 3), the form

$$dz = (ae^{i\omega} + be^{i\bar{\omega}})d\varphi \qquad \left(a = \frac{1 + (1-\tau_{\infty 2})^{-\beta}}{2\sqrt{2\alpha\tau_{\infty 2}}},\; b = \frac{1 - (1-\tau_{\infty 2})^{-\beta}}{2\sqrt{2\alpha\tau_{\infty 2}}}\right) \tag{1.5}$$

The complex pressure is given by the equation (reference 3)

$$Y + iX = \frac{\rho_0 V_{\infty 2}}{2}\int (e^{-i\bar{\omega}} - e^{-i\omega})d\varphi + p_0(1 - \tau_{\infty 2})^{\beta+1}\int d\bar{z} \tag{1.6}$$

2. The steady potential flow of a gas through an infinite cascade according to the well-known scheme of Kirchhoff with separations of the jet is next considered. The vanes of the cascade will be assumed to be plane (fig. 1).

Velocity of the gas in the flow at infinity is denoted by $V_{\infty 1}$ and its angle with the x-axis by $\theta_{\infty 1}$; velocity of the gas in the jet at infinity is denoted by $V_{\infty 2}$ and its angle with the

x-axis by $\theta_{\infty 2}$. The angle θ between the velocity vectors and the x-axis lies within the range $-2\pi < \theta \leqslant 0$.

The velocity field of the flow repeats itself for each displacement by the pitch of the cascade, that is, by the vector $he^{-i\lambda}$

The condition of constancy of the mass flow for steady flow of the gas gives

$$Q = -\rho_{\infty 1} V_{\infty 1} h \sin(\lambda + \theta_{\infty 1}) = \rho_{\infty 2} V_{\infty 2} n \tag{2.1}$$

where $\rho_{\infty 1}$ is the density of the gas at infinity in the flow, $\rho_{\infty 2}$ is the density of the gas at infinity in the jet, and n is the width of the gas jet at infinity.

By making use of expressions (1.1) for ρ, equation (2.1) can be represented in the form

$$-V_{\infty 1} h \left(1 - \frac{V_{\infty 1}^2}{2\alpha}\right)^{\beta} \sin(\lambda + \theta_{\infty 1}) = V_{\infty 2} n \left(1 - \frac{V_{\infty 2}^2}{2\alpha}\right)^{\beta} \tag{2.2}$$

The behavior of the function $f = \varphi + i\psi$ in the z-plane of the gas flow is now considered. For simplicity, the function f is assumed equal to zero at the critical point O of the flow (fig. 1). From the relation

$$d\varphi = \frac{\partial\varphi}{\partial s} ds = V_s ds$$

it follows that on moving along the streamline $\psi = 0$, the function φ varies monotonically from $-\infty$ at the point E (infinity in the flow) to $+\infty$ at the point C (infinity in the jet) and passes through the zero value at the critical point O where the streamline branches. The value of the potential $f = \varphi + i\psi$ at the critical point O' displaced by the period $he^{-i\lambda}$ relative to the point O is found and (fig. 1)

$$\varphi(O') = \int_{OMM'O'} V_s ds = \int_{OM} V_s ds + \int_{MM'} V_s ds + \int_{M'O'} V_s ds$$

where MM' is a cut parallel to the axis of the cascade.

The first and last integrals mutually cancel and therefore as MM' approaches infinity in the flow, the following equation is obtained:

$$\varphi(0') = V_{\infty 1} h \cos(\lambda + \theta_{\infty 1}) \tag{2.3}$$

Furthermore, from the relation

$$d\psi = \frac{dQ}{\rho_0}$$

where dQ is the quantity of gas flowing in unit time between infinitely near streamlines, it follows that on being displaced by the pitch of the cascade the function ψ receives an increment equal to Q/ρ_0. Hence,

$$\psi(0') = \frac{Q}{\rho_0} = -\left(1 - \frac{V_{\infty 1}^2}{2\alpha}\right)^{\beta} hV_{\infty 1} \sin(\lambda + \theta_{\infty 1}) \tag{2.4}$$

In this manner, the f-plane with double-sided cuts along the half-straight lines parallel to the axis of reals (fig. 2) corresponds to the region of the gas flow (fig. 1). All the cuts, because of the rule by which the cascade was constructed, are obtained from the initial one (the positive φ-axis) by simultaneous displacement along verticals and horizontals at distances that are multiples of Q/ρ_0 and $V_{\infty 1} h \cos(\lambda + \theta_{\infty 1})$, respectively.

Because $\omega = \theta + i\sigma$ is an analytic function of $f = \varphi + i\psi$, the problem may be solved by relating these functions with the aid of a parameter that varies in the upper semicircle of unit radius, as in the Levi-Civita method.

By considering the rectilinearity of the cuts in the f-plane, the analytic function $f(t)$ is found, which brings about the conformal transformation of the f-plane into the semicircle t. In the t-plane, the flow of an ideal fluid about the boundary of the semicircle is constructed. For this purpose, sources and vortices of strengths and intensities are located, as shown in figure 3, at the points $\bar{t}_\infty$ and $\bar{t}_\infty^{-1}$ that are symmetrical with respect to the circle and at the mirror reflection of these points in the diameter, that is, at the points $\bar{t}_\infty$ and t_∞^{-1}. At the origin of coordinates we place a sink of strength $2q$ (and a similar sink at infinity).

In the constructed flow in the t-plane, the upper semicircle of unit radius and the diameter of the semicircle are, of course, streamlines so that the stream function ψ maintains a constant value at the boundary of the upper semicircle t; in the $\bar{f}$-plane, this boundary will correspond to straight cuts. The complex potential of the constructed flow will have the form

$$f(t) = \frac{1}{2\pi i}\left[(\gamma + iq) \log (t - t_\infty) + (-\gamma + iq) \log\left(t - \frac{1}{\bar{t}_\infty}\right) + \right.$$

$$\left.(-\gamma + iq) \log (t - \bar{t}_\infty) + (\gamma + iq) \log\left(t - \frac{1}{\bar{t}_\infty}\right) - 2iq \log t\right] + \text{constant}$$

or

$$f(t) = \frac{1}{2\pi i}\left[(\gamma + iq) \log\left(t + \frac{1}{t} - 2M\right) - \right.$$

$$\left.(\gamma - iq) \log\left(t + \frac{1}{t} - 2\bar{M}\right)\right] + \text{constant} \qquad (2.5)$$

where γ is the intensity of the vortex and q is the strength of the sources, and

$$M = \frac{1}{2}\left(t_\infty + \frac{1}{t_\infty}\right)$$

The arbitrary constant in equation (2.5) is chosen so that $f(t)$ becomes zero at a certain point $t = e^{\epsilon i}$, the position of which will be subsequently determined.

Because the logarithm has multiple values, the upper semicircle of the t-plane will correspond to an f-plane with an infinite number of straight cuts $\psi = kq(k = 0, \pm 1, \pm 2, \ldots)$, where φ changes from $k\gamma$ to $+\infty$, as easily follows from equation (2.5) by substituting $t = e^{i\theta}$ (the arc of the semicircle) and $t = t_1$ where t_1 is the real amount of the interval (-1, +1), the diameter of the semicircle.

In this manner, the function (2.5) establishes a conformal mapping of the upper semicircle of the t-plane on the f-plane with the double-sided cuts represented in figure 2.

The function (2.5) is used in the work of N. I. Akhiezer (reference 2) where it is obtained by successive conformal mappings: the f-plane on the half plane, the half plane on the unit circle, and finally the circle on the upper semicircle.

The point $t = 0$ is carried by the transformation (2.5) into $f = +\infty$ so that the radii AC and BC go over into the infinite segments (figs. 2 and 3). The conformal property of the transformation breaks down at the points $t = -1$, $t = +1$, and $t = e^{i\epsilon}$ (the point $e^{i\epsilon}$ corresponds to the origin of the double-sided cut in the f-plane). The condition $df/dt = 0$ for $t = e^{i\epsilon}$ gives

$$\frac{\gamma + iq}{\cos \epsilon - M} = \frac{\gamma - iq}{\cos \epsilon - \bar{M}} \tag{2.6}$$

In order to obtain the elements of the motion, an expression for the derivative df/dt is required that is represented in the form

$$\frac{df}{dt} = \frac{q}{\pi} \frac{(t-e^{i\epsilon})(t-e^{-i\epsilon})(t-t^{-1})}{(t-t_\infty)(t-t_\infty^{-1})(t-\bar{t}_\infty)(t-\bar{t}_\infty^{-1})} \tag{2.7}$$

The quantities γ and q are next determined. When a point in the z-plane of the gas flow is displaced by the pitch of the cascade $he^{-i\lambda}$, the point t, corresponding to the point in the z-plane, goes over from one sheet of the Riemann surface to the next, passing once around the point E ($t = t_\infty$), as a result of which the function $f(t)$ receives an increment $\gamma + iq$, as follows from equation (2.5). Because the corresponding increments of the functions φ and ψ are equal to $V_{\infty 1} h \cos (\lambda + \theta_{\infty 1})$ and Q/ρ_0, respectively, the following equations are obtained:

$$\left.\begin{aligned} \gamma &= V_{\infty 1} h \cos (\lambda + \theta_{\infty 1}) \\ q &= -\left(1 - \frac{V_{\infty 1}^2}{2\alpha}\right)^{\beta} V_{\infty 1} h \sin (\lambda + \theta_{\infty 1}) \end{aligned}\right\} \tag{2.8}$$

From the expression for γ, it is evident, among other things, that $\gamma = 0$ corresponds to the case where the approaching flow has a velocity at infinity perpendicular to the axis of the cascade.

3. The function $\omega(t)$ is next determined. The function $\omega = \theta + i\sigma$ is regular within the semicircle t and has the following properties:

1. At the point $O(t = e^{i\epsilon})$, the real part of the function ω has a discontinuity, equal to π, because of the branching of the streamline. On the arc AO the angle θ is equal to $-\pi$ and on the arc OB it is equal to zero.

2. On the real diameter of the semicircle, the function $\omega(t) = \theta + i\sigma$ is real because its imaginary part $\sigma = 0$ on the free jets where $\tau = \tau_{\infty 2}$.

3. At the origin of coordinates $t = 0$, the function $\omega(t)$ is equal to $\theta_{\infty 2}$ because at infinity in the jet $\sigma = 0$ and $\theta = \theta_{\infty 2}$.

From the preceding discussion, it follows that the function $\omega(t)$ admits of analytical continuation in the lower semicircle and may be obtained by the Schwarz formula. Thus,

$$\omega(t) = \frac{1}{2\pi} \int_{|t| = 1} \theta \frac{e^{i\varphi} + t}{e^{i\varphi} - t} d\varphi = \frac{1}{2\pi} \int_{-\epsilon}^{\epsilon} - \pi \frac{e^{i\varphi} + t}{e^{i\varphi} - t} d\varphi = i \log \frac{e^{i\epsilon} - t}{1 - te^{i\epsilon}} \tag{3.1}$$

that branch of the logarithm being chosen that is equal to $i\epsilon$ for $t = 0$. If the third property is used from equation (3.1) for $t = 0$, it is found that $\epsilon = -\theta_{\infty 2}$. The value of t_∞ is obtained from equation (3.1) by making use of the value of the velocity at infinity in the stream

$$\omega(t_\infty) = \theta_{\infty 1} + i\sigma_{\infty 1} = i \log \frac{1 - t_\infty \exp i\theta_{\infty 2}}{\exp i\theta_{\infty 2} - t_\infty}$$

whence (reference 2)

$$\left.\begin{aligned} |t_\infty| &= \sqrt{\frac{\operatorname{ch}\sigma_{\infty 1} - \cos(\theta_{\infty 1} - \theta_{\infty 2})}{\operatorname{ch}\sigma_{\infty 1} - \cos(\theta_{\infty 1} + \theta_{\infty 2})}} \\ \arg t_0 &= \operatorname{arc\,tg} \frac{\operatorname{sh}\sigma_{\infty 1} \sin\theta_{\infty 2}}{\operatorname{ch}\sigma_{\infty 1} \cos\theta_{\infty 2} - \cos\theta_{\infty 1}} \end{aligned}\right\} \tag{3.2}$$

The pressure of the gas on a blade of the cascade is then computed. On the forward side of the plate, which is a streamline, the pressure is obtained by equation (1.6) where $V_{\infty 2}$ is the velocity in the stream.

The back side of the plate is in the gas at rest where the pressure is constant and equal to

$$p = p_0 (1 - \tau_{\infty 2})^{\beta+1}$$

If the fact that along a streamline $d\varphi = df$ is considered, a formula is obtained for the complex pressure in the form

$$Y + iX = \frac{\rho_0 V_{\infty 2}}{2} \int (e^{-i\bar{\omega}} - e^{-i\omega})\, df$$

or

$$Y - iX = \frac{\rho_0 V_{\infty 2}}{2} \int (e^{i\omega} - e^{i\bar{\omega}}) \frac{df}{dt}\, dt \tag{3.3}$$

where the integration is taken over the upper semicircle of the t-plane in the clockwise direction. By considering that after analytical continuation in the lower semicircle the function $\omega(t)$ assumes conjugate values at conjugate points, the following relation is obtained:

$$Y - iX = -\frac{\rho_0 V_{\infty 2}}{2} \int_{|t| = 1} e^{i\omega(t)} \frac{df}{dt}\, dt \tag{3.4}$$

where the integration is taken over the entire arc of the unit circle in the counterclockwise direction.

Substituting the value of df/dt from equation (2.7) in equation (3.4) gives

$$Y - iX = -\frac{\rho_0 q V_{\infty 2}}{2\pi} \int_{|t| = 1} e^{i\omega(t)} \frac{(t-e^{i\epsilon})(t-e^{-i\epsilon})(t-t^{-1})dt}{(t-t_\infty)(t-t_\infty^{-1})(t-\bar{t}_\infty)(t-\bar{t}_\infty^{-1})} \tag{3.5}$$

The function under the integral sign in equation (3.5) has three poles, $t = t_\infty$, $\bar{t} = t_\infty$, and $t = 0$, all of which lie within the unit circle. The residues of the function at these points are, respectively,

$$\frac{(t_\infty - e^{i\epsilon})(t_\infty - e^{-i\epsilon})}{(\bar{t}_\infty - \bar{t}_\infty)(t_\infty - \bar{t}_\infty^{-1})} \exp i\omega_{\infty 1}$$

$$\frac{(\bar{t}_\infty - e^{i\epsilon})(\bar{t}_\infty - e^{-i\epsilon})}{(\bar{t}_\infty - t_\infty)(\bar{t}_\infty - t_\infty^{-1})} \exp i\bar{\omega}_{\infty 1}$$

$$-\exp i\theta_{\infty 2}$$

From equation (2.6), however, it follows that

$$\frac{(t_\infty - e^{i\epsilon})(t_\infty - e^{-i\epsilon})}{(t_\infty - \bar{t}_\infty)(\bar{t}_\infty - \bar{t}_\infty^{-1})} = \frac{\gamma + iq}{2iq}$$

Hence, the residues may be represented in the form

$$\frac{\gamma + iq}{2iq} \exp(i\theta_{\infty 1} - \sigma_{\infty 1})$$

$$- \frac{\gamma - iq}{2iq} \exp(i\theta_{\infty 1} + \sigma_{\infty 1})$$

$$- \exp i\theta_{\infty 2}$$

From equation (3.5), applying the theorem on residues gives

$$Y - iX = - \frac{\rho_0 V_{\infty 2}}{2} \Big[- 2iq \exp i\theta_{\infty 2} +$$

$$(\gamma + iq) \exp(i\theta_{\infty 1} - \sigma_{\infty 1}) - (\gamma - iq) \exp(i\theta_{\infty 1} + \sigma_{\infty 1})\Big]$$

If the real and imaginary parts are separated,

$$Y = - \frac{\rho_0 V_{\infty 2}}{2} \Big[2q \sin \theta_{\infty 2} + (\gamma \cos \theta_{\infty 1} - q \sin \theta_{\infty 1}) \exp(- \sigma_{\infty 1}) -$$

$$(\gamma \cos \theta_{\infty 1} + q \sin \theta_{\infty 1}) \exp \sigma_{\infty 1}\Big] \qquad (3.6)$$

$$X = - \frac{\rho_0 V_{\infty 2}}{2} \Big[- 2q \cos \theta_{\infty 2} + (\gamma \sin \theta_{\infty 1} + q \cos \theta_{\infty 1}) \exp(- \sigma_{\infty 1}) -$$

$$(\gamma \sin \theta_{\infty 1} - q \cos \theta_{\infty 1}) \exp \sigma_{\infty 1}\Big] \qquad (3.7)$$

The velocity on the contour of the blade is assumed to remain finite; the total-pressure force on the blade will then be perpendicular to the velocity and therefore $X = 0$, that is,

$$(\gamma \sin \theta_{\infty 1} + q \cos \theta_{\infty 1}) \exp(-\sigma_{\infty 1}) -$$

$$(\gamma \sin \theta_{\infty 1} - q \cos \theta_{\infty 1}) \exp \sigma_{\infty 1} = 2q \cos \theta_{\infty 2} \tag{3.8}$$

Thus the pressure force of the gas on a blade of the cascade is by equations (3.6) and (2.8) equal to

$$Y = -\frac{\rho_0 V_{\infty 1} V_{\infty 2}}{2} h \left\{ -2\left(1 - \frac{V_{\infty 1}^2}{2\alpha}\right)^\beta \sin(\lambda + \theta_{\infty 1}) \sin \theta_{\infty 2} + \right.$$
$$\left[\cos(\lambda + \theta_{\infty 1}) \cos \theta_{\infty 1} + \left(1 - \frac{V_{\infty 1}^2}{2\alpha}\right)^\beta \sin(\lambda + \theta_{\infty 1}) \sin \theta_{\infty 1}\right] \exp(-\sigma_{\infty 1}) -$$
$$\left.\left[\cos(\lambda + \theta_{\infty 1}) \cos \theta_{\infty 1} - \left(1 - \frac{V_{\infty 1}^2}{2\alpha}\right)^\beta \sin(\lambda + \theta_{\infty 1}) \sin \theta_{\infty 1}\right] \exp \sigma_{\infty 1}\right\} \tag{3.9}$$

If in equation (3.9) β is set equal to 0 and the magnitude $\sigma_{\infty 1}$, determined by equations (1.4), is correspondingly replaced by

$$\sigma_{\infty 1} = \int_{\tau_{\infty 2}}^{\tau_{\infty 1}} \frac{d\tau}{2\tau} = \frac{1}{2} \log \frac{\tau_{\infty 1}}{\tau_{\infty 2}} = \log \frac{V_{\infty 1}}{V_{\infty 2}}$$

the formula for the pressure is obtained for the case of an ideal fluid (reference 2).

In order to determine the angle λ, entering equation (3.9), between the axis of the cascade and the x-axis, the ratio (2.6) and the values of γ and q from equations (2.8) are used. Thus

$$\operatorname{ctg}(\lambda + \theta_{\infty 1}) = i \left(1 - \frac{v_{\infty 1}^2}{2\alpha}\right)^\beta \frac{2 \cos \epsilon - (M + \bar{M})}{M - \bar{M}} \tag{3.10}$$

In order to compute the length of a cascade blade, equation (1.5) is used. Replacing $d\varphi$ by df gives

$$dz = \frac{q}{\pi}\left(a \frac{1 - te^{i\epsilon}}{e^{i\epsilon} - t} + b \frac{t - e^{i\epsilon}}{te^{i\epsilon} - 1}\right) \frac{(t - e^{i\epsilon})(t - e^{-i\epsilon})(t - t^{-1})dt}{(t - t_\infty)(t - \bar{t}_\infty)(t - t_\infty^{-1})(t - \bar{t}_\infty^{-1})}$$

This expression may be put in the form

$$dz = \frac{qa}{\pi} g_1(t)dt + \frac{qb}{\pi} g_2(t)dt \tag{3.11}$$

where

$$g_1(t) = \frac{e^{i\epsilon}(t-e^{-i\epsilon})^2(t-t^{-1})}{(t-t_\infty)(t-t_\infty^{-1})(t-\bar{t}_\infty)(t-\bar{t}_\infty^{-1})}$$

$$g_2(t) = \frac{e^{-i\epsilon}(t-e^{i\epsilon})^2(t-t^{-1})}{(t-t_\infty)(t-t_\infty^{-1})(t-t_\infty)(t-\bar{t}_\infty^{-1})}$$

The expansions of $g_1(t)$ and $g_2(t)$ into the sum of simple fractions are of the form

$$g_\nu(t) = \frac{A_\nu}{t-t_\infty} + \frac{B_\nu}{t-\bar{t}_\infty} + \frac{C_\nu}{t-t_\infty^{-1}} + \frac{D_\nu}{t-\bar{t}_\infty^{-1}} + \frac{E_\nu}{t} \qquad (\nu = 1,2)$$

where

$$A_1 = C_2 = \frac{\gamma+iq}{2iq} \exp(i\theta_{\infty 1} - \sigma_{\infty 1})$$

$$B_1 = D_2 = \frac{iq-\gamma}{2iq} \exp(i\theta_{\infty 1} + \sigma_{\infty 1})$$

$$E_1 = -e^{-i\epsilon}$$

$$C_1 = A_2 = \frac{\gamma+iq}{2iq} \exp(\sigma_{\infty 1} - i\theta_{\infty 1})$$

$$D_1 = B_2 = \frac{iq-\gamma}{2iq} \exp(- i\theta_{\infty 1} - \sigma_{\infty 1})$$

$$E_2 = -e^{i\epsilon}$$

These expressions for the coefficients are obtained if the following relations are used:

$$\exp i\omega_{\infty 1} = \frac{1-t_\infty e^{i\epsilon}}{e^{i\epsilon}-t_\infty}$$

$$\frac{(t_\infty-e^{i\epsilon})(t_\infty-e^{-i\epsilon})}{(t_\infty-\bar{t}_\infty)(t_\infty-\bar{t}_\infty^{-1})} = \frac{\gamma+iq}{2iq}$$

$$\exp i\bar{\omega}_{\infty 1} = \frac{1-\bar{t}_\infty e^{i\epsilon}}{e^{i\epsilon}-\bar{t}_\infty}$$

$$\frac{(\bar{t}_\infty-e^{-i\epsilon})(\bar{t}_\infty-e^{i\epsilon})}{(\bar{t}_\infty-t_\infty)(\bar{t}_\infty-t_\infty^{-1})} = \frac{iq-\gamma}{2iq}$$

If equation (3.11) is integrated over the upper semicircle in the t-plane in a counterclockwise direction and if relation $z(-1)-z(1) = l$ is used, the following expression for the length of a cascade blade is obtained:

$$l = \frac{qa}{\pi}\left[A_1 \lg \frac{-1-t_\infty}{1-t_\infty} + B_1 \lg \frac{-1-\bar{t}_\infty}{1-\bar{t}_\infty} + C_1 \lg \frac{-1-t_\infty^{-1}}{1-t_\infty^{-1}} + D_1 \lg \frac{-1-\bar{t}_\infty^{-1}}{1-\bar{t}_\infty^{-1}} + E_1 \lg (-1)\right] +$$

$$\frac{qb}{\pi}\left[A_2 \lg \frac{-1-t_\infty}{1-t_\infty} + B_2 \lg \frac{-1-\bar{t}_\infty}{1-\bar{t}_\infty} + C_2 \lg \frac{-1-t_\infty^{-1}}{1-t_\infty^{-1}} + D_2 \lg \frac{-1-\bar{t}_\infty^{-1}}{1-\bar{t}_\infty^{-1}} + E_2 \lg (-1)\right]$$

$$= \frac{qa}{\pi}\left[(A_1 + C_1)\lg\frac{t_\infty^{-1}+1}{t_\infty^{-1}} + (B_1 + D_1)\lg\frac{\bar{t}_\infty^{-1}+1}{\bar{t}_\infty^{-1}-1} + \right.$$

$$\left.(A_1 + B_1 + E_1)\lg(-1)\right] + \frac{qb}{\pi}\left[(A_2 + C_2)\lg\frac{t_\infty^{-1}+1}{t_\infty^{-1}-1} + \right.$$

$$\left.(B_2 + D_2)\lg\frac{\bar{t}_\infty^{-1}+1}{t_\infty^{-1}-1} + (A_2 + B_2 + E_2)\lg(-1)\right]$$

$$= \frac{qa}{\pi}\left[(A_1 + B_1 + C_1 + D_1)\lg\frac{R_1}{R_2} + \right.$$

$$\left. i\alpha\,(A_1 + C_1 - B_1 - D_1) + (A_1 + B_1 + E_1)\,\pi i\right] +$$

$$\frac{qb}{\pi}\left[(A_1 + B_1 + C_1 + D_1)\lg\frac{R_1}{R_2} + \right.$$

$$\left. i\alpha\,(A_1 + C_1 - B_1 - D_1) + (C_1 + D_1 + E_2)\,\pi i\right]$$

The magnitudes R_1, R_2, and α that enter this equation are shown in figure 4. Substituting the values of the coefficients and making use of equation (3.8) gives the following expression for the length of a blade:

$$l = + \frac{2q}{\pi}(a + b)\cos\theta_{\infty 2}\lg\frac{R_1}{R_2} +$$

$$\frac{\alpha}{\pi}(a + b)\Big[(\gamma\cos\theta_{\infty 1} - q\sin\theta_{\infty 1})\exp(-\sigma_{\infty 1}) +$$

$$(\gamma\cos\theta_{\infty 1} + q\sin\theta_{\infty 1})\exp\sigma_{\infty 1}\Big] +$$

$$\frac{a-b}{2}\Big[(\gamma\cos\theta_{\infty 1} - q\sin\theta_{\infty 1})\exp(-\sigma_{\infty 1}) -$$

$$(\gamma\cos\theta_{\infty 1} + q\sin\theta_{\infty 1})\exp\sigma_{\infty 1}\Big] + qa\sin\theta_{\infty 2}$$

The formula for the length of a blade in the case of an ideal fluid (reference 2) is obtained from the preceding equation for $\beta = 0$ if

$$a = \frac{1}{V_{\infty 2}}$$

$$b = 0$$

$$q = -V_{\infty 1}\ h \sin\ (\lambda + \theta_{\infty 1})$$

$$\gamma = V_{\infty 1}\ h \cos\ (\lambda + \theta_{\infty 1})$$

$$\sigma_{\infty 1} = \lg \frac{V_{\infty 1}}{V_{\infty 2}}$$

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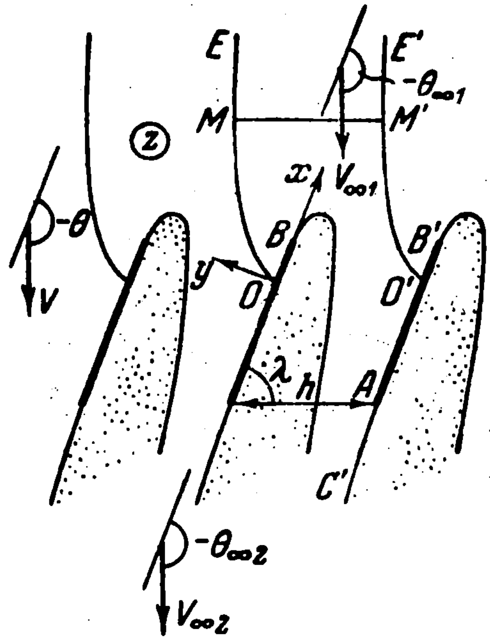

Figure 1.

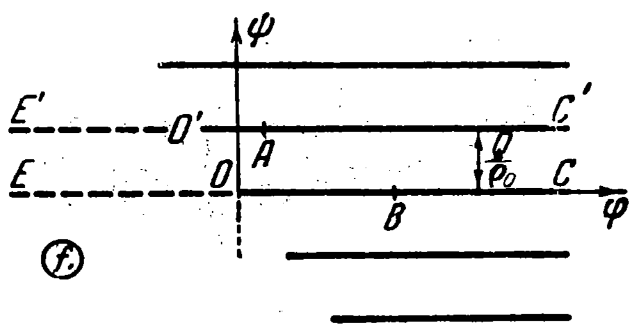

Figure 2.

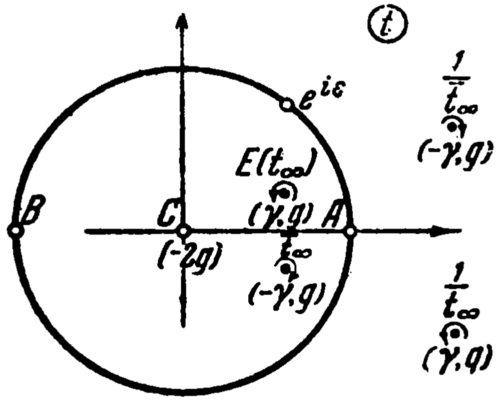

Figure 3.

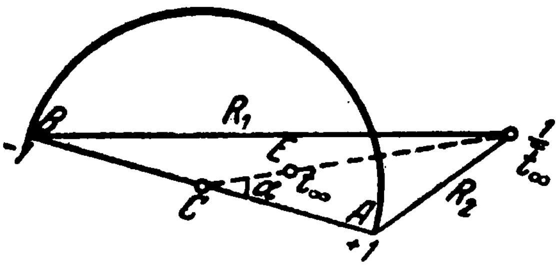

Figure 4.